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# ON BOUNDS AND EXACT VALUES OF $k$-EFFICIENT DOMINATION NUMBER OF A GRAPH

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Abstract. In a graph $G = (V, E)$, a set $S \subseteq V$ is an efficient dominating set if every vertex of $G$ is dominated exactly once by the vertices of $S$. As a generalization of this concept, $k$-efficient domination is introduced. A set $S$ is a $k$-efficient dominating set if there is a partition of $V$ which is a collection of $i$-neighbourhoods of vertices of $S$, where $i$'s vary between 0 and $k$. The minimum cardinality of a $k$- efficient dominating set is the $k$-efficient domination number of $G$, denoted by $\epsilon_k(G)$. In this paper, some bounds on $\epsilon_k(G)$ and exact values of $\epsilon_1(G)$ are obtained for products of paths and cycles.

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## 1. Introduction

Throughout $G = (V, E)$ is a connected graph with no loops, no parallel edges. In a graph $G$, the open neighbourhood of a vertex $v$ is the set $N(v) = \{u \in V : uv \in E\}$. The closed neighbourhood of $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. For any $S \subseteq V$, the open neighbourhood of $S$ is $N(S) = \cup_{x \in S} N(x)$ and the closed neighbourhood of $S$ is $N[S] = \cup_{x \in S} N[x]$. For any integer $k$, the closed $k$-neighbourhood of a vertex $v \in V$ is $N_k[v] = \{w \in V : d(v, w) \leq k\}$ and the open $k$-neighbourhood of a vertex $v$ is $N_k(v) = N_k[v] - \{v\}$. The degree of a vertex $v$, denoted by $deg(v)$ is the cardinality of $N(v)$. Further, $\delta(G) = min\{deg(v)|v \in V\}$ and $\Delta(G) = max\{deg(v)|v \in V\}$. For an integer $k \geq 1$, the $k$-degree of a vertex $v$ in $G$ is the cardinality of $N_k(v)$, denoted by $deg_k(v)$. The minimum $k$-degree of $G$ is $\delta_k(G) = min\{deg_k(v)|v \in V\}$ and the maximum $k$-degree of $G$ is $\Delta_k(G) = max\{deg_k(v)|v \in V\}$. The terms not defined here may be found in [3, 10].

A set $S \subseteq V$ is called an independent set if no two vertices in $S$ are adjacent. A set $D \subseteq V$ is called a dominating set if every vertex in $G$ is in $D$ or adjacent to a vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. More details on domination related parameters can be found in [1, 6]. The concept of *perfect d-dominating sets* [2] was first introduced by Biggs. Later in 1988, Bange et al. defined *efficient dominating sets* [1], which is same as the class of perfect 1-dominating sets. A dominating set $S$ is an efficient dominating set if for all $v \in V$, $|N[v] \cap S| = 1$. More results on efficient domination can be found in [5, 7, 8].

As a generalization of efficient domination, $k$-efficient domination was introduced by using partitions of $V$. For a positive integer $k$, a graph $G$ is said to be $k$-efficient[4] if there is a set $S = \{v_1, v_2, \ldots, v_t\} \subset V$ for which $\pi = \{N_k[v_1], N_k[v_2], \ldots, N_k[v_t]\}$ is a partition of $V$. The set $S$ is called as an exact $k$-efficient dominating set of $G$.

For a given positive integer $k$, every graph need not be $k$-efficient. For example, the graph $C_7$ is neither 1-efficient, nor 2-efficient. However for a given integer $k$, a partition of $V$ can be obtained by considering $i$-neighbourhood of some vertices of $G$ where $0 \leq i \leq k$.

**Definition 1.1.** [4] *A partition* $\pi = \{N_{i_1}[v_1], N_{i_2}[v_2], \ldots, N_{i_t}[v_t]\}$ *of* $V$ *is called a* $k$*-efficient partition if* $0 \leq i_j \leq k$ *for all* $j = 1, 2, \ldots, t$. *The vertices* $v_1, v_2, \ldots, v_t$ *are called the essential vertices of the partition* $\pi$ *and* $S = \{v_1, v_2, \ldots, v_t\}$ *is called a* $k$*-efficient dominating set in* $G$.

For any positive integer $k$, the $k$*-efficient domination number* of $G$ is the minimum cardinality of a $k$-efficient dominating set of $G$, denoted by $\epsilon_k(G)$.

**Observations 1.2.** *For any graph* $G$, $\epsilon_1(G) \geq \gamma(G)$.

In [4], some bounds for $\epsilon_k(G)$ in terms of order and degree are obtained. The exact values of $\epsilon_k(G)$ for some particular graphs like $P_2 \square P_n$, $P_3 \square P_n$ are determined. Further, in [4] it is proved that decision problems related to $\epsilon_1(G)$, $\epsilon_2(G)$ are NP-complete, even when restricted to bipartite graphs. This paper presents additional results to further develop the concept of $k$-efficient partition in graphs.

## 2. Bounds on $k$-efficient domination number

For any vertex $v$ in a graph $G$, eccentricity $e(v)$ is the distance from $v$ to a farthest vertex from $v$ in $G$. Further, radius $rad(G) = \min\{e(v) : v \in V\}$ and diameter $diam(G) = \max\{e(v) : v \in V\}$. In this section, a bound on $k$-efficient domination number is obtained in terms of the diameter of graph.

**Proposition 2.1.** *If* $S$ *is an exact* $k$*-efficient dominating set with at least two elements in a graph* $G$ *with* $rad(G) \geq k$, *then* $d(u, v) \geq 2k+1$ *for all* $u, v \in S$. *Further, for each* $u \in S$ *there exists* $v \in S$ *such that* $d(u, v) = 2k + 1$.

*Proof.* Let $S = \{v_1, v_2, \ldots, v_m\}$. Consider a shortest path $P$ between $v_i$ and $v_j$ for $1 \leq i, j \leq m$, $i \neq j$. Note that $N_k[v_i] \cap N_k[v_j] = \emptyset$. Consider vertices $x$ and $y$ in the path $P$ where $x \in N_k[v_i]$ and $y \in N_k[v_j]$ such that $d(v_i, x) = d(v_j, y) = k$. Now,

$$\begin{aligned} d(v_i, v_j) &= d(v_i, x) + d(x, y) + d(y, v_j) \\ &= 2k + d(x, y) \\ &\geq 2k + 1. \end{aligned}$$

Consider any vertex $u$ in $S$. Since $rad(G) \geq k$ and $S$ has at least two elements, $N_k[u] \subsetneq V$. Further, there exists a vertex $w \in N_k[u]$ and a vertex $x \in V \backslash N_k[u]$ such that $d(u, w) = k$ and $d(w, x) = 1$. Since $S$ is exact $k$-efficient dominating set, $x \in N_k[v]$ for some $v \in S$. If $d(v, x) < k$

then $d(v,w) \leq d(v,x) + d(x,w) \leq (k-1)+1 = k$. This implies $w \in N_k[v] \cap N_k[u] = \emptyset$, a contradiction. Therefore $d(v,x) = k$. Hence $d(u,v) = d(u,w) + d(w,x) + d(x,v) = 2k+1$. □

**Theorem 2.2.** *If* $\pi = \{N_{i_1}[v_1], \ldots, N_{i_l}[v_l]\}$ *is a* $k$*-efficient partition of* $V$*, then* $\deg_{i_1}(v_1) + \deg_{i_2}(v_2) + \cdots + \deg_{i_l}(v_l) = n - l$.

*Proof.* Since $\pi$ is $k$-efficient partition of $V$,

$$\begin{aligned} n = \Big|\bigcup_{j=1}^{l} N_{i_j}[v_j]\Big| = \sum_{j=1}^{l} \big|N_{i_j}[v_j]\big| &= \sum_{j=1}^{l} \big|N_{i_j}(v_j) \cup \{v_j\}\big| \\ &= \sum_{j=1}^{l} \big|N_{i_j}(v_j)\big| + \sum_{j=1}^{l} 1 \\ &= \sum_{j=1}^{l} \deg_{i_j}(v_j) + l \end{aligned}$$

□

**Theorem 2.3.** *If* $\pi = \{N_{i_1}[v_1], N_{i_2}[v_2] \ldots, N_{i_l}[v_l]\}$ *is a* $k$*-efficient partition of* $V$*, then*

$$diam(G) \leq \sum_{j=1}^{l} 2i_j + l - 1.$$

*Proof.* Note that for any $u, v \in N_{i_j}[v_j]$, $d(u,v) \leq d(u,v_j) + d(v_j,v) \leq 2i_j$. Let $x$ and $y$ be any two vertices in $V$ and $P$ be the shortest path between $x$ and $y$. Since $\pi$ is partition of $V$, it follows that $P \cap N_{i_j}[v_j] \neq \emptyset$ for at least one $j \in \{1,2,\ldots,n\}$. Without loss of generality (if necessary, then with suitable relabeling of $v_1, v_2, \ldots, v_n$) assume that $P \cap N_{i_j}[v_j] \neq \emptyset$, where $j = 1,2,\ldots,t$ for some $t \leq l$. Since $P$ is the shortest path and $\pi$ is partition of $V$, the vertices of the path may be considered as $x = x_{11}, x_{12}, \ldots, x_{1m_1}, x_{21}, x_{22}, \ldots, x_{2m_2}, \ldots, x_{t1}, x_{t2}, \ldots, x_{tm_t} = y$ in order, where $x_{rs} \in N_{i_r}[v_r]$ and $1 \leq r \leq t, 1 \leq s \leq m_r$. Now,

$$\begin{aligned} d(x,y) &= d(x_{11}, x_{1m_1}) + d(x_{1m_1}, x_{21}) + d(x_{21}, x_{2m_2}) + \ldots + d(x_{t1}, x_{tm_t}) \\ &= \sum_{p=1}^{t} d(x_{p1}, x_{pm_p}) + \sum_{p=1}^{t-1} d(x_{pm_p}, x_{(p+1)1}) \\ &\leq \sum_{j=1}^{t} 2i_j + \sum_{p=1}^{t-1} 1 \\ &= \sum_{j=1}^{t} 2i_j + (t-1) \\ &\leq \sum_{j=1}^{l} 2i_j + (l-1). \end{aligned}$$

Since $x, y$ are arbitrary, it follows that $diam(G) \leq \sum\limits_{j=1}^{l} 2i_j + l - 1$. □

In Theorem 2.3, by replacing number of essential vertices by $\epsilon_k(G)$, we get the following result.

**Corollary 2.4.** *If* $\pi = \{N_{i_1}[v_1], N_{i_2}[v_2], \dots, N_{i_l}[v_l]\}$ *be a* $k$*-efficient partition of* $V$*, then*

$$diam(G) \leq \epsilon_k(G) - 1 + \sum_{j=1}^{l} 2i_j.$$

**Corollary 2.5.** *For a connected non-trivial graph* $G$*,*

$$\epsilon_k(G) \geq \left\lceil \frac{diam(G)+1}{2k+1} \right\rceil.$$

*Proof.* Follows by the fact that $i_j \leq k\ \forall j$ in Corollary 2.4. □

## 3. $k$-Efficient Domination Number of Cylindrical Grid Graphs

For any two graphs $G$ and $H$, the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \square H)$ such that $(u_1, v_1)(u_2, v_2) \in E(G \square H)$, whenever $v_1 = v_2$ and $u_1 u_2 \in E(G)$, or $u_1 = u_2$ and $v_1 v_2 \in E(H)$. If $u_1, u_2, \dots, u_m$ are the vertices of $P_m$ and $w_1, w_2, \dots, w_n$ are the vertices of $C_n$, then the vertex $(u_i, w_j)$ of $P_m \square C_n$ is represented by $v_{i,j}$ where $1 \leq i \leq m$, $1 \leq j \leq n$. The vertices $v_{i,1}$, $v_{i,2}$, $\dots$, $v_{i,n}$ are considered as $i^{th}$ row vertices, whereas the vertices $v_{1,j}$, $v_{2,j}$, $\dots$, $v_{m,j}$ are considered as $j^{th}$ column vertices of $P_m \square C_n$.

**Proposition 3.1.** *For the graph* $P_2 \square C_n$*, there exists an efficient dominating set if and only if* $n$ *is a multiple of 4.*

*Proof.* Suppose $S = \{v_{1,i_1}, v_{1,i_2}, \dots, v_{1,i_s}, v_{2,j_1}, v_{2,j_2}, \dots, v_{2,j_t}\}$ is an efficient dominating set in $P_2 \square C_n$. Let $v_{1,i_\alpha}, v_{1,i_\beta} \in S$. Without loss of generality assume that $1 < i_\alpha < i_\beta < n$. Since $S$ is an efficient dominating set, $i_\beta - i_\alpha \geq 3$. Assume $i_\beta - i_\alpha = 3$. Then $N[v_{1,i_\alpha}] = \{v_{1,(i_\alpha-1)}, v_{1,i_\alpha}, v_{1,(i_\alpha+1)}, v_{2,i_\alpha}\}$ and $N[v_{1,i_\beta}] = \{v_{1,(i_\beta-1)}, v_{1,i_\beta}, v_{1,(i_\beta+1)}, v_{2,i_\beta}\}$. Further, for any vertex $v_{2t} \in S$, $N[v_{2t}] \cap N[v_{1i_\alpha}] = \emptyset$ and $N[v_{2,t}] \cap N[v_{1,i_\beta}] = \emptyset$. Therefore either $t < i_\alpha - 1$ or $t > i_\beta + 1$. Then $v_{2,(i_\alpha+1)}, v_{2,(i_\alpha+2)} \notin N[x]$ for any $x \in S$. This contradicts that the set $S$ is an efficient dominating set. Thus we must have $i_\beta - i_\alpha \geq 4$. Therefore for any two vertices $x, y \in S$ lying in the same copy of $C_n$, $d(x,y) \geq 4$. Let $v_{1,i_\alpha}, v_{2,i_\beta} \in S$ be vertices such that $d(v_{1,i_\alpha}, v_{2,i_\beta})$ is least. Then $d(v_{1,i_\alpha}, v_{2,i_\beta}) \geq 3$. Otherwise, $N[v_{1,i_\alpha}] \cap N[v_{2,i_\beta}] \neq \emptyset$. Assume $d(v_{1,i_\alpha}, v_{2,i_\beta}) > 3$. Note that $N[v_{1,i_\alpha}] = \{v_{1,(i_\alpha-1)}, v_{1,i_\alpha}, v_{1,(i_\alpha+1)}, v_{2,i_\alpha}\}$, $N[v_{2,i_\beta}] = \{v_{2,(i_\beta-1)}, v_{2,i_\beta}, v_{2,(i_\beta+1)}, v_{1,i_\beta}\}$ and for any vertex $v_{2,t} \in S$, $N[v_{2,t}] \cap N[v_{1,i_\alpha}] = \emptyset$. Further if $t \neq i_\beta$, then $N[v_{2,t}] \cap N[v_{2,i_\beta}] = \emptyset$. Therefore either $t < i_\alpha - 1$ or $t > i_\beta + 1$. Then $v_{2,(i_\alpha+1)} \notin N[x]$ for any $x \in S$, which is a contradiction. Thus $d(v_{1,i_\alpha}, v_{2,i_\beta}) = 3$. Hence it follows that if $x, y$ are the nearest essential vertices from different copies of $C_n$, then $d(x,y) = 3$. Now suppose $v_{1,i_\alpha} \in S$ is arbitrary, then by the above argument, we get $S = \{v_{1,(i_\alpha +_n 4l)}, v_{2,(i_\alpha +_n (4l+2)):\ l \in \mathbb{Z}}\}$, where $+_n$ represents addition modulo $n$. This shows that $v_{1,j} \in S$ if and only if $j = i_\alpha +_n 4l$ for some $l \in \mathbb{Z}$. Note that $v_{1,(n +_n i_\alpha)} = v_{1,i_\alpha} \in S$. Thus $n$ is a multiple of 4.

Conversely, if $n \equiv 0 (\mathrm{mod}\, 4)$, then $S = \{v_{1,1}, v_{1,5}, v_{1,9}, \dots, v_{1,(n-3)}, v_{2,3}, v_{2,7},$

$v_{2,11}, \ldots, v_{2,(n-1)}\}$ is an efficient dominating set in $P_2 \Box C_n$. This completes the proof. $\square$

**Theorem 3.2.** *For any $n \geq 3$,*

$$\epsilon_1(P_2 \Box C_n) = \begin{cases} (n+3)/2 & \text{if } n \text{ is odd} \\ n/2 & \text{if } n \text{ is a multiple of } 4 \\ (n+6)/2 & \text{otherwise.} \end{cases}$$

*Proof.* Consider the following cases.
**Case (1):** $n$ is a multiple of 4.
In [9], it has been proved that $\gamma(P_2 \Box C_n) = n/2$, when $n$ is a multiple of 4. By Observations 1.2, $\epsilon_1(P_2 \Box C_n) \geq \gamma(P_2 \Box C_n) = n/2$. To prove the equality, we need to prove that $\epsilon_1(P_2 \Box C_n) \leq n/2$. Let $\pi_1 = \{N_1[v_{1,4i+1}], N_1[v_{2,4i+3}] : 0 \leq i \leq n\}$. Then $\pi_1$ is a 1-efficient partition of $V(P_2 \Box C_n)$ of cardinality $n/2$. Thus $\epsilon_1(P_2 \Box C_n) \leq n/2$.
**Case (2):** $n$ is not a multiple of 4.
Let $D$ be any minimum efficient partition set. Let $r$ denote the number of essential vertices with 0-neighbourhood. Let $D_1$ be the set of essential vertices lying in the first copy of $C_n$ with 1-neighbourhood and $D_2$ be that in the second copy of $C_n$. Let $|D_1| = l_1$ and $|D_2| = l_2$. Note that each vertex of $D_1$ dominates 3 vertices in the first copy of $C_n$, while each vertex of $D_2$ dominates one vertex of the first copy of $C_n$. Therefore,

$$3l_1 + l_2 \leq n. \tag{1}$$

By similar argument for second copy,

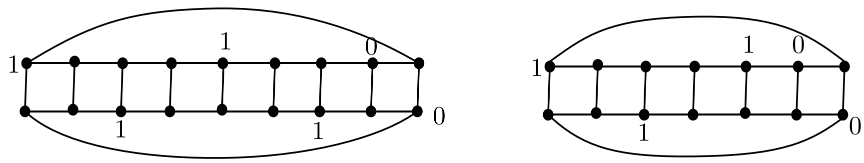
$$l_1 + 3l_2 \leq n. \tag{2}$$

Therefore $l_1 + l_2 \leq \lfloor \frac{n}{2} \rfloor$.
**Sub-Case (i):** $n \equiv 1 (\text{mod } 4)$.
Then the partition set $\pi_1$ does not dominate the vertices $v_{1,n-1}$ and $v_{2,n}$. Hence the set $\pi_2 = \pi_1 \bigcup \{N_0[v_{1,n-1}], N_0[v_{2,n}]\}$ is a 1-efficient partition of $V(P_2 \Box C_n)$. Thus $|D| \leq \frac{n+3}{2}$. Since $n$ is odd, $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Now $r \geq 2n - 4\lfloor \frac{n}{2} \rfloor = 2n - 4\left(\frac{n-1}{2}\right) = 2$. Suppose $l_1 + l_2 < \lfloor \frac{n}{2} \rfloor$. Then $l_1 + l_2 = \lfloor \frac{n}{2} \rfloor - m$, for some $m \geq 1$ and $r = 2n - 4\left(\frac{n-1}{2} - m\right) = 2 + 4m$. Then $|D| = l_1 + l_2 + r = \frac{n-1}{2} - m + 2 + 4m = \frac{n+3}{2} + 3m > \frac{n+3}{2}$, which is a contradiction. Thus $l_1 + l_2 = \frac{n-1}{2}$, $r = 2$ and hence $|D| = \frac{n+3}{2}$.
**Sub-Case (ii):** $n \equiv 3 (\text{mod } 4)$.
Then the partition set $\pi_1$ is not dominating $v_{2,n-1}$ and $v_{2,n}$. Hence the set $\pi_2' = \pi_1 \bigcup \{N_0[v_{2,n-1}], N_0[v_{2,n}]\}$ is a 1-efficient partition of $V(P_2 \Box C_n)$. As in Sub-case (i), we get $|D| = \frac{n+3}{2}$.
**Sub-Case (iii):** $n \equiv 2 (\text{mod } 4)$. Now $\pi_3 = \pi_2 \bigcup \{N_0[v_{1,n-2}], N_0[v_{2,n-1}]\}$ is a 1-efficient partition of $V(P_2 \Box C_n)$. Thus $|D| \leq (n+6)/2$. Let $n = 4l + 2$. Then $\frac{n}{2} = 2l + 1$ is odd. Suppose $l_1 + l_2 = \frac{n}{2}$. Since $4(l_1 + l_2) + r = 2n$, we get $r = 0$ and hence inequalities 1 and 2 become equations. Since $\frac{n}{2}$ is odd, without loss of generality, assume that $l_1 > l_2$. Then $l_1 = l_2 + m$ for some $m \geq 1$. By Substituting $l_1 = l_2 + m$ in these equations, we get $m = 0$, which is a contradiction. Thus $l_1 + l_2 < \frac{n}{2}$. Then $l_1 + l_2 = \frac{n}{2} - t$ for some

$t \geq 1$. Since $4(l_1 + l_2) + r = 4\left(\frac{n}{2} - t\right) + r = 2n$, we get $r = 4t$. Then $|D| = l_1 + l_2 + 4t = \frac{n}{2} - t + 4t = \frac{n}{2} + 3t \geq \frac{n}{2} + 3 = \frac{n+6}{2}$.

□

**Remark 3.3.** *Examples for choosing a 1-efficient dominating sets are given in Figure 1, Figure 2 and Figure 3.*



$n \equiv\ 1\ (mod\ 4)$ $n \equiv\ 3\ (mod\ 4)$

FIGURE 1

**Lemma 3.4.** *For any $n \geq 3$, every 1-efficient dominating set of $P_3 \square C_n$ contains at least one vertex of each column.*

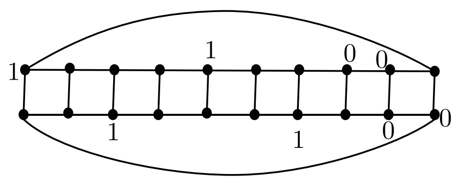
*Proof.* For any $i$ with $1 \leq i \leq n$, suppose that $i^{th}$ column of $P_3 \square C_n$ has no essential vertex. Then the vertices of $i^{th}$ column must be dominated by 3 distinct vertices. Then by Pigeon-Hole Principle, either $(i-1)^{th}$ or $(i+1)^{th}$ column contains at least two essential vertices. Then a vertex of either $(i-1)^{th}$ or $(i+1)^{th}$ column is dominated by two essential vertices, which is a contradiction. □

**Theorem 3.5.** *For any $P_3 \square C_n$ cylindrical grid graph $\epsilon_1(P_3 \square C_n) = n$ except for $n = 4, 7$. Further $\epsilon_1(P_3 \square C_4) = 5$ and $\epsilon_1(P_3 \square C_7) = 8$.*

*Proof.* By Lemma 3.4, $\epsilon_1(P_3 \square C_n) \geq n$. It can be observed that $\epsilon_1(P_3 \square C_n) = n+1$ for $n = 4, 7$. For $n \notin \{4, 7\}$, it remains to show that 1-efficient partition of cardinality $n$ exists.

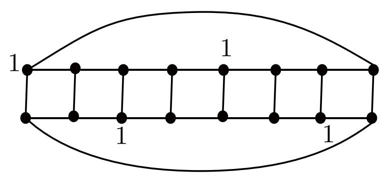
If $n \equiv 0 (\text{mod } 3)$, then the partition of $V$ given by $\{N_1[v_{1,2}], N_1[v_{1,5}], \ldots, N_1[v_{1,n-1}], N_0[v_{2,1}], N_0[v_{2,4}], \ldots, N_0[v_{2,n-2}], N_1[v_{3,3}], N_1[v_{3,6}], \ldots, N_1[v_{3,n}]\}$ is a 1-efficient partition with cardinality $n$.

If $n \equiv 1 (\text{mod } 3)$, then the partition of $V$ given by $\{N_0[v_{1,1}], N_0[v_{1,3}], N_1[v_{1,5}], N_1[v_{1,8}], \ldots, N_1[v_{1,n-5}], N_1[v_{1,n-1}], N_1[v_{2,2}], N_0[v_{2,6}], N_0[v_{2,9}], \ldots, N_0[v_{2,n-7}], N_1[v_{2,n-3}], N_1[v_{3,4}], N_1[v_{3,7}], \ldots, N_1[v_{3,n-6}], N_0[v_{3,n-4}], N_0[v_{3,n-2}], N_0[v_{3,n}]\}$ is a 1-efficient partition with cardinality $n$.



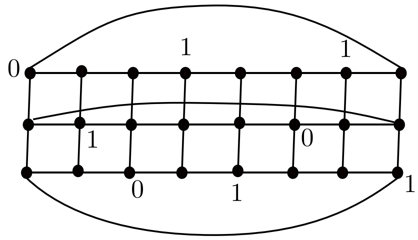
$n \equiv\ 2\ (mod\ 4)$

FIGURE 2



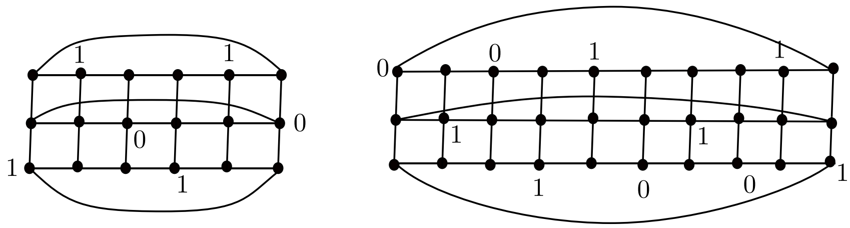
$n \equiv\ 0\ (mod\ 4)$

FIGURE 3




$n \equiv\ 2\ (mod\ 3)$

FIGURE 4




$n \equiv\ 0(mod\ 3)$ $n \equiv\ 1(mod\ 3)$

FIGURE 5

If $n \equiv 2(\text{mod } 3)$, then $\{N_0[v_{1,1}], N_1[v_{1,4}], N_1[v_{1,7}], \ldots, N_1[v_{1,n-1}], N_1[v_{2,2}], N_0[v_{2,6}], N_0[v_{2,9}], \ldots, N_0[v_{2,n-2}], N_0[v_{3,3}], N_1[v_{3,5}], N_1[v_{3,8}], \ldots, N_1[v_{3,n}]\}$ is a 1-efficient partition with cardinality $n$. □

Example for 1-efficient dominating set in $(P_3 \Box C_n)$ is given in Figure 4 and 5.

## Conclusion

In this paper, some bounds on $k$-efficient domination number in terms of diameter and $k$ value are obtained. The existence of efficient dominating set in $P_2 \Box C_n$ is discussed. Further 1-efficient domination numbers of $P_2 \Box C_n$ and $P_3 \Box C_n$ are determined.

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